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Multiplicity of solutions for a class of superlinear non-local fractional equations

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In this paper, we are concerned with the problem driven by a non-local integro-differential operator with homogeneous Dirichlet boundary conditions. As a particular case, we study multiple solutions for the following non-local fractional Laplace equations:

\[
\begin{align*}
(−\Delta)^s u &= f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega ,
\end{align*}
\]

where \(s \in (0, 1)\) is fixed parameter, \(\Omega\) is an open bounded subset of \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\) \((n > 2s)\) and \((−\Delta)^s\) is the fractional Laplace operator. By a variant version of the Mountain Pass Theorem, a multiplicity result is obtained for the above-mentioned superlinear problem without Ambrosetti–Rabinowitz condition. Consequently, the result may be looked as a complete extension of the previous work of Wang and Tang to the non-local fractional setting.

**Keywords:** fractional Laplacian; integrodifferential operator; Mountain Pass Theorem; superlinear condition

**AMS Subject Classifications:** 35A15; 35J91; 49J35

1. Introduction

In this article, we are interested in the following non-local fractional equations:

\[
\begin{align*}
(−\Delta)^s u &= f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega ,
\end{align*}
\]

where \(s \in (0, 1)\) is fixed parameter, \(\Omega\) is an open bounded subset of \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\), \(n > 2s\) and \((−\Delta)^s\) is the fractional Laplace operator. In [1], Di Nezza et al. showed that the fractional Laplacian \((−\Delta)^s\) can be viewed as a non-local pseudo-differential operator and \((−\Delta)^s\) becomes \(-\Delta\) as \(s \to 1^−\), see [1] for the details.

In recent years, more and more attention has been devoted to the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and for concrete

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applications in many fields such as optimization, finance, phase transition phenomena and so on, see [1–3] and the references therein. This is one of the reasons why non-local fractional problems are widely studied in the literature, see [4–13,16–18] and the references therein for non-local fractional Laplacian equations with superlinear or asymptotically linear and subcritical or critical non-linearities. In [8], Servadei and Valdinoci studied the existence of non-trivial Mountain Pass-type solutions for equations driven by a non-local integro-differential operation with homogeneous Dirichlet boundary conditions. In [18], Autuori and Pucci studied the existence and multiplicity of solutions for elliptic equations involving the fractional Laplacian in \( \mathbb{R}^n \). In [19], Felmer et al. proved the existence of positive solutions for non-linear Schrödinger equations involving the fractional Laplacian in \( \mathbb{R}^n \). Using Ricceri’s critical points theorems, Ferrara et al. [20,21] investigated the multiplicity of solutions for perturbed non-local fractional problems with certain values of the parameters. Similar variational methods can be used to study various problems, see for example also [22–25].

There are many interesting problems in the standard framework of the Laplacian, widely studied in the literature. A natural question is whether or not the existence results obtained in this classical context can be extended to the non-local framework of the fractional Laplacian operators. Zhou [26] showed the existence of solutions of asymptotically linear Dirichlet problem for the \( p \)-Laplacian via a variant version of Mountain Pass Theorem. Using the same method, Wang and Tang [27] obtained the existence and multiplicity of solutions for a class of superlinear \( p \)-Laplacian equations.

Motivated by their works, we consider the following non-local problem with homogeneous Dirichlet boundary conditions investigated by Servadei et al. [7] and the following works [8–15]:

\[
\begin{align*}
-\mathcal{L}_K u &= f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align*}
\]  

(1.2)

where \( \mathcal{L}_K \) is the integro-differential operator defined as follows:

\[
\mathcal{L}_K u(x) := \frac{1}{2} \int_{\mathbb{R}^n} \left( u(x + y) + u(x - y) - 2u(x) \right) K(y) \, dy, \quad x \in \mathbb{R}^n,
\]  

(1.3)

with the kernel \( K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty) \) such that

\begin{align*}
(K_1) \quad & mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1|; \\
(K_2) \quad & \Lambda > 0 \text{ such that } K(x) \geq \Lambda |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}; \\
(K_3) \quad & K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}.
\end{align*}

A prototype for \( K \) is given by the singular kernel \( K(x) = |x|^{-(n+2s)} \) which leads to the fractional Laplace operator \( (-\Delta)^{s} \), which, up to normalization factors, may be defined as

\[
(-\Delta)^s u(x) := -\frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n.
\]  

(1.4)

Obviously, the corresponding fractional equation in the above model (1.4) becomes problem (1.1).
By a weak solution of problem (1.1), we mean a weak solution of the following problem:

$$\begin{cases}
\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x - y)dxdy - \int_{\Omega} f(x, u(x))\varphi(x)dx = 0, \quad \forall \varphi \in X_0 \\
u \in X_0,
\end{cases}$$

(1.5)

where $X_0$ is a Hilbert space that will be defined in Section 2. Hence, a weak solution $u \in X_0$ of problem (1.1) is defined as follows:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))dxdy = \int_{\Omega} f(x, u(x))\varphi(x)dx,$$

(1.6)

for any $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. Note that $X_0 \subset H^s(\mathbb{R}^n)$, see [8, Lemma 5].

Let $f(x, 0) = 0$ and $F(x, t) = \int_0^t f(x, s)ds$. Moreover, suppose that the non-linearity $f$ satisfy the following conditions:

1. $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is subcritical in $t$, i.e. there is a constant $q \in (2, 2^*)$, $2^* := 2n/(n - 2s)$ such that

$$\lim_{t \to \infty} \frac{f(x, t)}{|t|^{q-1}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

2. There exists $\theta \geq 1$ such that $\theta \mathcal{H}(x, t) \geq \mathcal{H}(x, \sigma t)$ for $(x, t) \in \Omega \times \mathbb{R}$ and $\sigma \in [0, 1]$, where $\mathcal{H}(x, t) = f(x, t)t - 2F(x, t)$.

3. $f(x, t)t \geq 0$ for any $x \in \overline{\Omega}$, $t \in \mathbb{R}$ and $\lim_{|t| \to \infty} \frac{f(x, t)}{t} = +\infty$ uniformly in $x \in \overline{\Omega}$.

4. $\limsup_{t \to 0} \frac{f(x, t)}{t} = p(x)$ uniformly in $x \in \overline{\Omega}$, where $p \in L^\infty(\overline{\Omega})$ satisfies $p(x) \leq \lambda_1$ for all $x \in \overline{\Omega}$ and $p(x) < \lambda_1$ on some $\Omega_0 \subset \Omega_1$ with $|\Omega_0| > 0$, where $\Omega_1 := \{x \in \Omega : e_1(x) \neq 0\}$.

Here, $\lambda_1 > 0$ that has an associated eigenfunction $e_1$ is the first eigenvalue of $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, see Section 2 for more details. The main result of the paper is the following theorem.

**Theorem 1.1** Let $K$ satisfy the assumptions $(K_1)$–$(K_3)$ and $f$ satisfy the assumptions $(f_1)$–$(f_4)$. Then the problem (1.5) has at least two non-trivial weak solutions in $X_0$ in which one is non-negative and another is non-positive.

Particularly, we can deduce the following multiplicity result for the fractional Laplacian, here $\Omega_1 = \Omega$ in condition $(f_4)$, see Corollary 8 in Servadei and Valdinoci [14].

**Corollary 1.1** Let $f$ satisfy the assumptions $(f_1)$–$(f_4)$. Then, the problem (1.6) has at least two non-trivial weak solutions in $H^s(\mathbb{R}^n)$ in which one is non-negative and another is non-positive.

In [8], Servadei and Valdinoci obtained an existence theorem for problem (1.5) using the Mountain Pass Theorem under Ambrosetti–Rabinowitz condition, which is originally due to Ambrosetti and Rabinowitz in [28], that is, there exist $\mu > 2$, $M > 0$ such that
(AR) \[ 0 < \mu F(x, t) \leq tf(x, t), \quad \text{for all} \quad t \in \mathbb{R}, \quad |t| \geq M \quad \text{and a.e.} \quad x \in \Omega. \]

A lot of works concerning superlinear elliptic boundary value problem have been done using the (AR) condition. It is easy to see that the role of (AR) is to ensure the boundedness of the Palais–Smale sequences of the energy functional. From (AR) condition, it easily follows that \( \lim_{|t| \to \infty} f(x, t)/t = +\infty \) which implies that the problem (1.1) is superlinear at infinity. However, there are many functions which are superlinear at infinity, but does not satisfy the condition (AR). Indeed, from condition (AR) it follows that for some \( c, \ d > 0 \)

\[
F(x, t) \geq c|t|^\mu - d \quad \text{for} \quad (x, t) \in \Omega \times \mathbb{R}.
\]  

(1.7)

Clearly, the function

\[ f(x, t) = 2t \log(1 + |t|) \]  

(1.8)

does not satisfy (1.7). Therefore, many efforts have been made to overcome the difficulties due to the absence of the (AR) condition, see for example [29,30] and the references therein. The assumption \((f_2)\) originally introduced by Jeanjean [31] has been always employed to prove the existence and multiplicity of solutions for superlinear Laplacian equations, see [32,33] and the references therein. Using Morse theory, Ferrara et al. [34] studied the existence and multiplicity results of weak solutions for non-local fractional problems under the assumption \((f_2)\), see [34] for further details.

In [26,35], Zhou and Li studied the existence of positive solutions for Laplacian equations under the following monotonicity condition:

\[ f(x, t)/t \text{ is nondecreasing in } |t|. \]  

(1.9)

Liu and Li [36] proved that condition (1.9) implies condition \((f_2)\). Moreover, we can find some examples that fulfil condition \((f_2)\) but not condition (1.9). For instance, the function

\[ f(x, t) = 5t \log(1 + t^2) + 9 \sin t \]

satisfies condition \((f_2)\), but does not satisfy condition (1.9) as \( \theta = 100 \), see [27].

The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about our working space. In Section 3, we prove a sequence of lemmas in order to apply a variant version of Mountain Pass Theorem. In Section 4, we give the proofs of the main results.

2. Preliminaries

In this section, we give some preliminary results which will be used in the sequel.

2.1. Variational framework

We first briefly recall the related definition and notes for functional space \(X_0\) introduced in [7].

The functional space \(X\) denotes the linear space of Lebesgue measurable functions from \(\mathbb{R}^n\) to \(\mathbb{R}\) such that the restriction to \(\Omega\) of any function \(g\) in \(X\) belongs to \(L^2(\Omega)\) and the map \((x, y) \mapsto (g(x) - g(y))\sqrt{K(x - y)}\) is in \(L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (C\Omega \times C\Omega), dx dy)\)
(here $C\Omega := \mathbb{R}^n \setminus \Omega$). Also, we denote by $X_0$ the following linear subspace of $X$

$$X_0 := \{ g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$ 

Note that $X$ and $X_0$ are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [7, Lemma 5.1]. Moreover, the space $X$ is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left( \int_Q |g(x) - g(y)|^2 K(x - y) dx \, dy \right)^{1/2},$$

(2.1)

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Omega$ and $O = (C\Omega) \times (C\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$.

We can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \left( \int_Q |v(x) - v(y)|^2 K(x - y) dx \, dy \right)^{1/2}.$$

(2.2)

as norm on $X_0$. Also, $(X_0, \| \cdot \|_{X_0})$ is a Hilbert space, with scalar product

$$\langle u, v \rangle_{X_0} := \int_Q (u(x) - u(y)) (v(x) - v(y)) K(x - y) dx \, dy.$$ 

(2.3)

Notice that in (2.3) (and in the related scalar product), the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

Denote by $H^s(\Omega)$ the usual fractional Sobolev space with respect to the Gagliardo norm

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dx \, dy \right)^{1/2}.$$ 

(2.4)

Now, we should present some facts which will be used later.

**Lemma 2.1** (see [8, Lemma 5])  If $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions $(K_1)$–$(K_3)$, then $X \subset H^s(\Omega)$ and $X_0 \subset H^s(\mathbb{R}^n)$.

**Lemma 2.2** (see [8, Lemma 6])  Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions $(K_1)$–$(K_3)$. Then

(i) there exists a positive constant $C$, depending only on $n$ and $s$, such that for any $u \in X_0$

$$\|u\|_{L^{2^*_s}(\Omega)}^2 \leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx \, dy.$$

(ii) there exists a positive constant $C > 1$, depending only on $n$, $s$, $\Lambda$ and $\Omega$, such that for any $u \in X_0$

$$\|v\|_{X_0} \leq \|u\|_X \leq C \|v\|_{X_0}.$$

**Lemma 2.3** (see [8, Lemma 8])  Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions $(K_1)$–$(K_3)$. Then the embedding $j : X_0 \hookrightarrow L^{q'}(\mathbb{R}^n)$ is continuous for any $q \in [1, 2^*)$, while it is compact whenever $q \in [1, 2^*)$. 

2.2. An eigenvalue problem for $-\mathcal{L}_K$

In [11], Servadei and Valdinoci investigated the eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the eigenvalue of the problem

$$\begin{cases}
-\mathcal{L}_K u = \lambda u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}$$

(2.5)

More precisely, the following weak formulation of (2.5) was discussed:

$$\begin{cases}
\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x - y)dxdy = \lambda \int_{\Omega} u(x)\varphi(x)dx, \\
u \in X_0.
\end{cases}$$

(2.6)

We call that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\mathcal{L}_K$ if there exists a non-trivial solution $u \in X_0$ of problem (2.5) or its weak formulation (2.6), and any solution will be called an eigenfunction corresponding to the eigenvalue $\lambda$.

**Lemma 2.4** (see [11, Proposition 9]) Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions $(K_1)$–$(K_3)$. Then

(a) problem (2.6) admits an eigenvalue $\lambda_1 > 0$ that is simple and has an associated eigenfunction which is non-negative function $e_1 \in X_0$ with $\|e_1\|_{L^2(\Omega)} = 1$;

(b) problem (2.6) has a set of the eigenvalues which consists of a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ with

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots \text{ and } \lambda_k \to +\infty \text{ as } k \to +\infty.$$

Moreover, for any $k \in \mathbb{N}$, the eigenvalues can be characterized as follows:

$$\lambda_{k+1} = \min_{u \in \mathbb{P}_k \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y)dxdy}{\int_{\Omega} |u(x)|^2dx},$$

where $\mathbb{P}_k := \{u \in X_0 : \langle u, e_j \rangle_{X_0} = 0, \forall j = 1, \ldots, k\}$.

3. Some lemmas

First, we observe that problem (1.2) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows:

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y)dxdy - \int_{\Omega} F(x, u(x))dx.$$

It is well known that the functional $\mathcal{J}$ is Frechét differentiable in $X_0$ and for any $\varphi \in X_0$

$$\langle \mathcal{J}'(u), \varphi \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x - y)dxdy - \int_{\Omega} f(x, u(x))\varphi(x)dx.$$ 

Thus, critical points of $\mathcal{J}$ are solutions of problem (1.2).
In order to look for a critical point of $J$, we shall use the following variation of the Mountain Pass Theorem.

**Proposition 3.1** (see Schechter [37]) Let $Y$ be a real Banach space with its dual space $Y^*$ and suppose that $J \in C^1(Y, \mathbb{R})$ satisfies the condition

$$\max\{J(0), J(u_1)\} \leq \rho < \beta \leq \inf_{\|u\| = \alpha} J(u)$$

(3.1)

for some $\rho < \beta$, $\alpha > 0$ and $u_1 \in Y$ with $\|u_1\| > \alpha$. Let $c$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq \gamma \leq 1} J(\gamma(\eta)),$$

where $\Gamma = \{\gamma \in C([0, 1], Y) : \gamma(0) = 0; \gamma(1) = u_1\}$ is the set of continuous paths joining 0 and $u_1$. Then there exists a subsequence $\{u_n\} \subset Y$ such that

$$J(u_n) \to c \geq \beta \quad \text{as} \quad n \to \infty,$$

(1 + $|u_n|$) $\|J'(u_n)\|_{Y^*} \to 0 \quad \text{as} \quad n \to \infty.$$

(3.2)

**Lemma 3.1** If $(f_4)$ holds, then there exists a constant $\rho \in (0, 1)$ such that

$$\int_\Omega p(x)|u|^2 dx < \rho \|u\|_{X_0}^2$$

(3.3)

for each $u \in X_0$.

**Proof** Otherwise, by contradiction, there exists a sequence $\{u_n\} \subset X_0$ such that

$$\int_\Omega p(x)|u_n|^2 dx \geq \left(1 - \frac{1}{n}\right) \|u_n\|_{X_0}^2.$$

(3.4)

Set $v_n = \|u_n\|_{X_0}^{-1} u_n$, it follows that

$$\int_\Omega p(x)|v_n|^2 dx \geq \left(1 - \frac{1}{n}\right).$$

(3.5)

By Lemma 2.4 and $(f_4)$, we have

$$\int_\Omega p(x)|v_n|^2 dx \leq \lambda_1 \int_\Omega |v_n|^2 dx \leq \|v_n\|_{X_0}^2 = 1.$$

(3.6)

Combining (3.5) and (3.6), we obtain

$$\left(1 - \frac{1}{n}\right) \leq \int_\Omega p(x)|v_n|^2 dx \leq \lambda_1 \int_\Omega |v_n|^2 dx \leq \|v_n\|_{X_0}^2 = 1.$$

(3.7)

Since $\{v_n\}$ is bounded in $X_0$, up to a subsequence, Lemma 2.3 gives that

$$v_n \rightharpoonup v \text{ in } X_0, \quad v_n \to v \text{ in } L^2(\Omega).$$

(3.8)
Letting $n \to \infty$ in (3.7), we get
\begin{align}
\lambda_1 \int_{\Omega} |v|^2 \, dx &= \lim_{n \to \infty} \|v_n\|_{X_0}^2 = 1, \quad (3.9) \\
\int_{\Omega} (p(x) - \lambda_1)|v|^2 \, dx &= 0. \quad (3.10)
\end{align}

From (3.9), together the weakly lower semicontinuity of the norm with the variational characterization of $\lambda_1$, we have
\begin{equation}
1 = \lambda_1 \int_{\Omega} |v|^2 \, dx \leq \|v\|_{X_0}^2 \leq \liminf_{n \to \infty} \|v_n\|_{X_0}^2 = 1. \quad (3.11)
\end{equation}

Then it follows that
\begin{equation}
1 = \lambda_1 \int_{\Omega} |v|^2 \, dx = \|v\|_{X_0}^2. \quad (3.12)
\end{equation}

From (3.12) and Lemma 2.4, we know that $v$ is the eigenfunction corresponding to the first eigenvalue of the problem (2.5). From (3.10) and (f4), we get that $(p(x) - \lambda_1)v^2(x) = 0$ a.e. on $\Omega$. And then, $(p(x) - \lambda_1)v^2(x) = 0$ a.e. on $\Omega_1$. Finally, we obtain that $p(x) = \lambda_1$ a.e. on $\Omega_1$, which contradicts with (f4). The proof is thus complete. \( \square \)

**Lemma 3.2** Let $f$ satisfy the conditions (f1) and (f4). Then there exist $\alpha > 0$ and $\beta > 0$ such that for any $u \in X_0$ with $\|u\|_{X_0} = \alpha$ we have that $J(u) \geq \beta$.

**Proof** Let $\epsilon > 0$ be small enough such that $\rho + \epsilon/\lambda_1 < 1$, where $\rho$ is the same as that in Lemma 3.1. By (f1) and (f4), there exist $\delta_1, \delta_2 > 0$ and $M > 0$ such that
\begin{align*}
F(x, t) &\leq \frac{1}{2}(p(x) + \epsilon)|t|^2, \quad \forall|t| < \delta_1, x \in \Omega, \\
F(x, t) &\geq \frac{1}{2}\epsilon|t|^q, \quad \forall|t| > \delta_2, x \in \Omega, \\
F(x, t) &\leq Mt^2 \leq M\delta_1^{2-q}|t|^q, \quad \forall \delta_1 \leq |t| \leq \delta_2, x \in \Omega.
\end{align*}
(3.13)

where $q$ is the same as in (f4). Taking $R = \max\{1/(2\epsilon), M\delta_1^{2-q}\}$, we have
\begin{equation}
F(x, t) \leq \frac{1}{2}(p(x) + \epsilon)|t|^2 + R|t|^q, \quad \text{for all} \ (x, t) \in \Omega \times \mathbb{R}. \quad (3.14)
\end{equation}

In view of assumption ($K_2$), Lemmas 2.2, 2.3 and 3.1, we obtain
\begin{align}
J(u) &\geq \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{2}\int_{\Omega} (p(x) + \epsilon)|u|^2 \, dx - R\|u\|_{L^q(\Omega)}^q \\
&\geq \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{2}\rho\|u\|_{X_0}^2 - \frac{\epsilon}{\lambda_1}\|u\|_{X_0}^2 - C_1\|u\|_{L^q(\Omega)}^q \\
&\geq \left(1 - \frac{\epsilon}{\lambda_1}\right)\|u\|_{X_0}^2 - C_2\|u\|_{X_0}^q.
\end{align}
(3.15)

where $C_1, C_2 > 0$ are constants. Now, let $\|u\|_{X_0} = \alpha$. Since $1 - \rho - \epsilon/\lambda_1 > 0$ and $2 < q$, we can choose $\alpha$ sufficiently small such that
\[
\frac{1}{2}
\left(1 - \rho - \frac{\varepsilon}{\lambda_1}\right)\alpha^2 - C_2\alpha^q := \beta > 0.
\]
This completes the proof. \(\square\)

**Lemma 3.3** Let \(f\) satisfy the conditions \((f_1)\) and \((f_3)\). Then there exists \(u_0 \in X_0\) such that \(\|u_0\|_{X_0} > \alpha\) and \(J(u_0) < 0\).

**Proof** By means of \((f_3)\), we have that \(\lim_{t \to +\infty} f(x, t)/t = +\infty\) and \(f(x, t) \geq 0\) as \(t \geq 0\). Then for any \(\varepsilon > 0\), there is some \(N > 0\) such that \(f(x, t)/t \geq 1/\varepsilon\) for all \(t > N\) and \(x \in \overline{\Omega}\). Set \(c(\varepsilon) = (1/\varepsilon)N\), it follows that
\[
f(x, t) \geq \frac{1}{\varepsilon} - c(\varepsilon),
\]
for each \(t \geq 0\) and \(x \in \overline{\Omega}\), which means that
\[
f(x, rt) \geq \frac{1}{\varepsilon} - cr - c(\varepsilon)t,
\]
for all \(t \geq 0\), \(x \in \overline{\Omega}\) and \(0 \leq r \leq 1\). Integrating \((3.17)\) with respect to \(r\) on \([0, 1]\), we have
\[
F(x, t) \geq \frac{1}{2\varepsilon}t^2 - c(\varepsilon)t
\]
for all \(t \geq 0\). Then we have
\[
F(x, te_1) \geq \frac{1}{2\varepsilon}t^2e_1^2 - c(\varepsilon)te_1,
\]
where \(e_1 \geq 0\) is an eigenfunction corresponding to the first eigenvalue of the operator \(-\mathcal{L}_K\) with homogeneous Dirichlet boundary data, see Lemma 2.4 for further details. Dividing by \(t^2\), we have
\[
\frac{F(x, te_1)}{t^2} \geq \frac{1}{2\varepsilon}e_1^2 - c(\varepsilon)e_1/t,
\]
Then, we obtain
\[
\int_{\Omega} \frac{F(x, te_1)}{t^2} dx \geq \int_{\Omega} \left(\frac{1}{2\varepsilon}e_1^2 - c(\varepsilon)e_1/t\right) dx.
\]
Letting \(t \to +\infty\), we deduce
\[
\lim_{t \to +\infty} \int_{\Omega} \frac{F(x, te_1)}{t^2} dx \geq \int_{\Omega} \frac{1}{2\varepsilon}e_1^2 dx,
\]
for all \(\varepsilon > 0\). Since \(\varepsilon\) is arbitrary, we obtain by letting \(\varepsilon \to 0\)
\[
\lim_{t \to +\infty} \int_{\Omega} \frac{F(x, te_1)}{t^2} dx = +\infty.
\]
Therefore,
\[
\frac{J(te_1)}{t^2} = \frac{1}{2}\|e_1\|^2_{X_0} - \int_{\Omega} \frac{F(x, te_1)}{t^2} dx \to -\infty,
\]
as \( t \to +\infty \). Hence, let \( t_0 \) be sufficiently large and \( u_0 = t_0e_1 \), then we have \( J(u_0) < 0 \).

\[ \square \]

4. Proof of main results

In this section, we are in a position to prove Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1

Define

\[ \Gamma = \{ \gamma \in C([0, 1], X_0) : \gamma(0) = 0; \gamma(1) = u_0 \}, \quad c = \inf_{\gamma \in \Gamma \atop 0 \leq \theta \leq 1} \max_{\gamma(\theta)}. \quad (4.1) \]

According to Proposition 3.1, from Lemmas 3.2 and 3.3 we know that there exists a sequence \( \{u_n\} \subset X_0 \) such that

\[ J(u_n) \to c, \quad (1 + ||u_n||_{X_0})||J'(u_n)||_{X_0^*} \to 0, \quad \text{as } n \to \infty. \quad (4.2) \]

Then, it follows that

\[ c + o(1) = J(u_n) - \frac{1}{2}\langle J'(u_n), u_n \rangle = \int_{\Omega} \left( \frac{1}{2}f(x, u_n)u_n - F(x, u_n) \right)dx. \quad (4.3) \]

In the following, let us show that the sequence \( \{u_n\} \) is bounded in \( X_0 \). Otherwise, there exists a subsequence of \( \{u_n\} \) (still denoted by \( \{u_n\} \)) satisfying \( ||u_n||_{X_0} \to \infty \) as \( n \to \infty \). Let \( v_n = ||u_n||_{X_0}^{-1}u_n \), then \( ||v_n||_{X_0} = 1 \). By Lemma 2.3, up to a subsequence, we have

\[ v_n \to v \text{ in } X_0, \quad v_n \to v \text{ in } L^q(\mathbb{R}^n), \quad v_n \to v \text{ a.e. } x \in \mathbb{R}^n. \quad (4.4) \]

If \( v \equiv 0 \), as in [31], we can choose a sequence \( \{t_n\} \subset [0, 1] \) such that \( J(t_nu_n) = \max_{t \in [0, 1]} J(tu_n) \). For any positive integer \( m \), we can choose \( r = 2\sqrt{m} \) such that \( r||u_n||_{X_0}^{-1} \in (0, 1) \) as \( n \) large enough. By (4.4) and (3.14), it easily follows that

\[ \lim_{n \to \infty} \sup \int_{\Omega} F(x, rv_n)dx \leq 0. \quad (4.5) \]

Hence, for \( n \) large enough, (4.5) yields

\[ J(t_nu_n) \geq J(r||u_n||_{X_0}^{-1}u_n) = J(rv_n) = 2m - \int_{\Omega} F(x, rv_n)dx \geq m, \]

which implies that \( J(t_nu_n) \to +\infty \). But \( J(0) = 0, J(u_n) \to c \), so \( t_n \in (0, 1) \) and

\[ \langle J'(t_nu_n), t_nu_n \rangle = t_n \frac{d}{dt} \bigg|_{t=t_n} J(tu_n) = 0. \]

Then, from \((f_2)\) it follows that

\[ \frac{1}{\theta}J(t_nu_n) = \frac{1}{\theta} \left( J(t_nu_n) - \frac{1}{2}\langle J'(t_nu_n), t_nu_n \rangle \right) \]

\[ = \frac{1}{2\theta} \int_{\Omega} \mathcal{H}(x, t_nu_n)dx \]

\[ \leq \frac{1}{2} \int_{\Omega} \mathcal{H}(x, u_n)dx \]

\[ = J(u_n) - \frac{1}{2}\langle J'(u_n), u_n \rangle \to c. \quad (4.6) \]
This contradicts the fact that $\mathcal{J}(t_n u_n) \to +\infty$.

If $v \not\equiv 0$, then the set $\Omega' := \{x \in \Omega : v(x) \neq 0\}$ has positive Lebesgue measure and $|u_n(x)| \to +\infty$ on $\Omega'$. Notice that (4.2) gives

$$||u_n||_{X_0}^2 - \int_{\Omega} f(x, u_n)u_n dx = \langle \mathcal{J}'(u_n), u_n \rangle = o(1).$$

By (f3), we know that $f(x, u)u \geq 0$, and hence

$$1 - o(1) = \int_{\Omega} \frac{f(x, u_n)u_n}{||u_n||_{X_0}^2} dx = \left( \int_{\Omega'} + \int_{\Omega' \setminus \Omega} \right) \frac{f(x, u_n)u_n}{||u_n||^2} |v_n|^2 dx \geq \int_{\Omega'} \frac{f(x, u_n)u_n}{||u_n||^2} |v_n|^2 dx.$$

Then, Fatou’s lemma and (f2) imply that

$$1 \geq \lim inf_{n \to \infty} \int_{\Omega'} \frac{f(x, u_n)u_n}{||u_n||^2} |v_n|^2 dx = +\infty.$$

This is a contradiction. Thus, $\{u_n\}$ is bounded in $X_0$.

Similar to the proof of Proposition 12 in [8], by (3.14) and Lemma 2.3, it is easy to show that any bounded sequence $\{u_n\}$ in $X_0$ such that $\sup \{||\mathcal{J}'(u_n), \varphi|| : \varphi \in X_0, \|\varphi\|_{X_0} = 1\} \to 0$ has a convergent subsequence. Thus, we know that $\{u_n\}$ have a subsequence which converges to a weak solution $u \in X_0$ of problem (1.1).

Next, let us consider the following truncated problem:

$$\begin{cases}
-L_K u = f_+(x, u) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

where

$$f_+(x, u) = \begin{cases} f(x, u) & t \geq 0 \\
0 & t \leq 0. \end{cases}$$

Obviously, $f_+(x, u)$ satisfies the conditions (f1) – (f4). Thus, there is a non-trivial weak solution $u^+ \in X_0$ of problem (4.8) and it is also a non-trivial weak solution of problem (1.1) by the definition of $f_+$. Similar to the proof of Corollary 13 in [8], it is easy to prove that $u^+$ is non-negative in $\mathbb{R}^n$.

Finally, let us consider another truncated problem:

$$\begin{cases}
-L_K u = f_-(x, u) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

where

$$f_-(x, u) = \begin{cases} 0 & t \geq 0 \\
f(x, u) & t \leq 0. \end{cases}$$

with the same arguments, it is easy to show that there is a non-positive weak solution $u^- \in X_0$ of problem (4.10) and it is also a non-positive weak solution of problem (1.1) by the definition of $f_-$.

**Proof of Corollary 1.1** The desired conclusion immediately follows from Theorem 1.1 by taking $K(x) = |x|^{-(n+2s)}$ and by recalling that $X_0 \subseteq H^s(\mathbb{R}^n)$, see Lemma 5 in Servadei and Valdinoci [8].
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